JOINT ENTROPY

$$H(X,Y) = \sum_i \sum_j P_{ij} \log_2 \left(\frac{1}{P_{ij}} \right)$$

$$P_{ij} = Pr(X = s_i, Y = r_j)$$

$H(X,Y) = 3.375$ bits

From example.

Marginal Probabilities.

$$Pr(X = s_0) = Pr(X = s_1)$$

$$= Pr(X = s_2) = Pr(X = s_3)$$

$$= \frac{1}{4}.$$

Entropy of X is,

$$H(X) = H\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

4 symbols with probability $= \frac{1}{4}$ each

$$H(X) = 4 \times \frac{1}{4} \log_2 4$$

$$= 1 \times 2 = 2 \text{ bits.}$$

$$H(Y) = H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

$$= \frac{1}{2} \log_2 2 + \frac{1}{4} \log_2 4$$

$$+ \frac{1}{8} \log_2 8 + \frac{1}{8} \log_2 8$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{3}{8}$$

$$= 1 + \frac{3}{4} = 1.75 \text{ bits.}$$

$H(X,Y) = 3.375 > H(X) = 2$

$H(X,Y) > H(Y) = 1.75$

$\left\{ \begin{array}{l} H(X,Y) \geq H(X) \\ H(X,Y) \geq H(Y) \end{array} \right.$

Joint Entropy or information in X & Y

$\geq$ information in either X or Y.

$$H(X) + H(Y) = 2 + 1.75$$

$$= 3.75 > H(X,Y) = 3.375$$

$$H(X,Y) \leq H(X) + H(Y)$$

Joint Entropy or Total Information is lower or equal to sum of entropies of individual sources.

$$H(X,Y) = \sum_i \sum_j Pr(s_i, r_j) \log_2 \frac{1}{Pr(s_i, r_j)}$$

$$H(X) = \sum_i Pr(s_i) \log_2 \frac{1}{Pr(s_i)}$$

$$= \sum_i \sum_j Pr(s_i, r_j) \log_2 \frac{1}{Pr(s_i)}$$

$$= Pr(s_i)$$ ← From total Prob rule.

$$H(Y) = \sum_j Pr(r_j) \log_2 \frac{1}{Pr(r_j)}$$

$$= \sum_j \sum_i Pr(s_i, r_j) \log_2 \left(\frac{1}{Pr(r_j)}\right)$$

$$H(X,Y) - H(X) - H(Y)$$

$$= \sum_i \sum_j \underbrace{P_r(s_i, r_j) \log_2 \left(\frac{1}{P_r(s_i, r_j)} \right)}_{H(X,Y)}$$

$$- \sum_i \sum_j \underbrace{P_r(s_i, r_j) \log_2 \left(\frac{1}{P_r(s_i)} \right)}_{-H(X)}$$

$$- \sum_i \sum_j \underbrace{P_r(s_i, r_j) \log_2 \left(\frac{1}{P_r(r_j)} \right)}_{H(Y)}.$$

$$= \sum_i \sum_j Pr(s_i, r_j) \left\{ \log_2 \frac{1}{Pr(s_i, r_j)} - \log_2 \frac{1}{Pr(s_i)} - \log_2 \frac{1}{Pr(r_j)} \right\}$$

$$= \underbrace{\sum_i \sum_j Pr(s_i, r_j) \log_2 \left(\frac{Pr(s_i) Pr(r_j)}{Pr(s_i, r_j)} \right)}_{H(X,Y) - H(X) - H(Y)}$$

$$= \sum_{i,j} Pr(s_i, r_j) \log_2 \left(\frac{Pr(s_i) Pr(r_j)}{Pr(s_i, r_j)} \right)$$

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$$\sum_i p_i \log_2 x_i$$

$$\leq \log_2 \cdot \sum_i p_i x_i$$

Jensen's Inequality
Since $\log_2 x$ is concave.

$$p_i \geq 0$$

$$\sum p_i = 1$$

$$\sum_{i,j} P_H(s_i, r_j) \log_2 \left(\frac{P_H(s_i) P_H(r_j)}{P_H(s_i, r_j)} \right)$$

$$\leq \log_2\left(\sum_{i,j} \Pr(s_i, r_j)\, \frac{\Pr(s_i)\Pr(r_j)}{\Pr(s_i, r_j)}\right)$$

$$= \log_2\left(\sum_i \sum_j \Pr(s_i)\Pr(r_j)\right)$$

$$= \log_2\left(\underbrace{\left(\sum_i \Pr(s_i)\right)}_{1}\underbrace{\left(\sum_j \Pr(r_j)\right)}_{1}\right)$$

$$= \log_2 1 = 0 .$$

$$\boxed{H(X,Y) - H(X) - H(Y) \leq 0}$$

Jensen's Inequality

$$H(X,Y) - H(X) - H(Y) \leq 0$$

$$\Rightarrow \boxed{H(X,Y) \leq H(X) + H(Y)}$$